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Quantum Structures in Macroscopic Reality

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We show that it is possible to construct macroscopic entities that entail a quantum logical structure. We do this by means of the introduction of a simple macroscopic entity and study its structure in terms of lattices and graphs, and show that the lattice is non-Boolean.

1. INTRODUCTION

The principal idea that we want to put forward in this paper is the following: some of the most characteristic properties of quantum entities can be found as properties of ordinary mechanistic macroscopic entities as well. Two groups of researchers (D. Aerts, T. Durt, and B. Van Bogaert in Brussels, and A. A. Grib and R. R. Zapatrin in St. Petersburg) have been investigating this idea, and have approached it from different directions. In this joint paper we investigate the connection between the results of the two groups.

The mechanistic macroscopic entity that we will present in this section has been presented in Aerts (1985, 1986, 1987) with the aim of giving a possible explanation for the nonclassical character of the quantum probability model. In this earlier work it is shown that a lack of knowledge about the change that the experiments exert on the physical system gives rise to a nonclassical probability model, isomorphic to a quantum probability model. But an explicit construction of the lattice of properties of the example has not yet been made.

Although quantum logic was derived from the study of the lattices of properties of quantum entities, few of the different approaches make it

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possible to construct explicitly the lattice of properties for a concrete physical situation of a physical entity and experiments on this physical entity. Some approaches do make this kind of construction possible, and it is one of these physically founded approaches, namely the Geneva approach (Piron, 1976, 1990; Aerts, 1981, 1982), that we will use to construct explicitly the lattice of properties connected to our physical example. Other approaches, more of a mathematical nature, give mathematical algorithms to construct the lattice. If an orthogonality relation is defined on the set of states, the lattice can be constructed by the procedure of the "biorthogonal" (Birkhoff, 1967; Zapatrin, 1988; Finkelstein and Finkelstein, 1983). We will investigate this construction on our concrete physical situation. Let us introduce the example.

2. DESCRIPTION OF THE MACROSCOPIC ENTITY

We will give a detailed description of the macroscopic entity and the set of experiments that we consider on the entity.

The physical entity S that we consider is a particle with fixed negative charge q such that it can be located on a sphere of radius r at a point $v = (r, \theta, \phi)$. With every vector v of the three-dimensional vector space of space directions with origin at the center of the sphere, there corresponds a state p_v of the entity.

The experiment e_u consists of the following operation: we choose two particles with positive charges q_1 and q_2 such that $q_1 + q_2 = Q$. The charge q_1 is chosen at random in the interval $[0, Q]$. This choice represents the lack of knowledge about the experimental situation. Once the charges q_1 and q_2 are chosen we put the two particles diametrically on the sphere, such that q_1 is at the point $u = (r, \alpha, \beta)$ and q_2 is at the point $-u = (r, \pi - \alpha, \pi + \beta)$. This is the

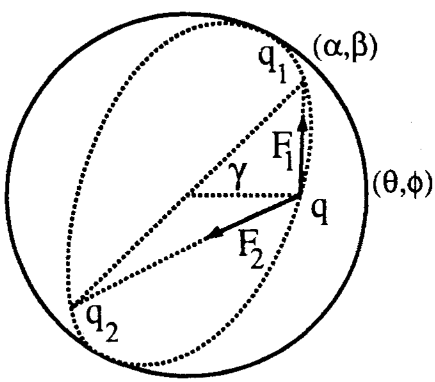

Fig. 1

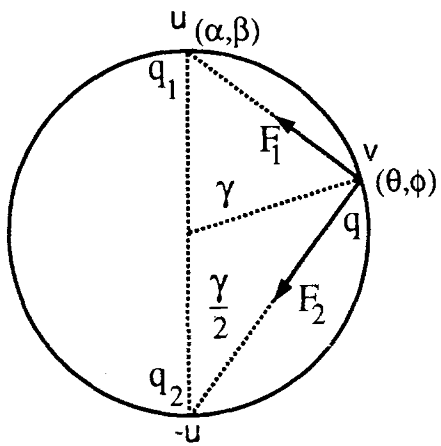

Fig. 2

setup of the experiment e_u, the entity being in state p_v. Under the influence of the Coulomb forces F_1 (between q and q_1) and F_2 (between q and q_2) (Figure 1), the charge q will move, and we suppose that this happens in a viscous medium such that finally, by means of friction, the charge q will end up at q_1 or q_2. If it ends up at q_1 (q_2) we give the outcome 1_u (2_u) for the experiment e_u, which changes the state p_v into one of the states p_u or p_{-u} (Figure 2).

3. QUANTUM LOGICAL STRUCTURE

Let us investigate the quantum logical structure of this situation. We consider two specific points v and u of the sphere and their diametrically opposed points $-v$ and $-u$. With these points correspond four possible states, p_v, p_{-v}, p_u, and p_{-u}, and two possible experiments, e_v and e_u. The state p_u (p_{-u}) is an "eigenstate" of the experiment e_u with "eigenoutcome" 1_u (2_u) [by which we mean that if the entity is in state p_u (p_{-u}), the experiment e_u gives outcome 1_u (2_u) with probability equal to 1]. Indeed, state p_u is the only state in which the experiment e_u never can give the outcome 2_u. Even if q_1 is uncharged, and the total charge is concentrated in q_2, the Coulomb force between q_2 and q is on the radius of the sphere, and hence q is in an "unstable" equilibrium with regard to this Coulomb force, and will not be moved by it. The state p_v (p_{-v}) is an "eigenstate" of the experiment e_v with "eigenoutcome" 1_v (2_v).

If the entity is in state p_u (p_v) and the experiment e_v (e_u) is performed, then the two outcomes 1_v (1_u) and 2_v (2_u) can occur. Indeed, since the amount of charge that finally is absorbed by particle q_1 or particle q_2 of experiment e_u is not determined (this is exactly the lack of knowledge about the complete

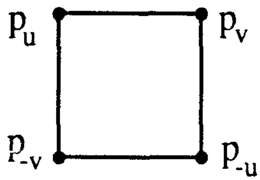

Fig. 3

reality of the situation), even if v is close in space to u, the experiment e_u can have an outcome 2_u, and the state p_v will then be changed to p_{-u}. If the entity is in state p_{-v} (p_u) and the experiment e_u (e_v) is performed, then also the two outcomes 1_u and 2_u (1_v and 2_v) can occur. The states p_v (p_u) and p_{-v} (p_{-u}) are orthogonal states.

The specification of this dynamics of measurements and changes of states makes it possible to draw the graph in Figure 3 (Finkelstein and Finkelstein, 1983; Grib and Zapatrin, 1990), which describes different situations, not only corresponding to Figures 1 and 2. Two points of the graph, representing two states, are connected by a vertex if there exists an experiment that transforms one state into the other one. We can see that in our example orthogonal states are not connected by a vertex.

3.1. Direct Construction of the Lattice of Properties

As we mentioned, we shall now use the Geneva approach to construct explicitly the lattice of properties connected to the example, and we shall see that this construction naturally leads to a non-Boolean lattice. We introduce the necessary concepts, and then immediately make the construction for our example.

1. *Introduction of the concept of property.* A property a, as introduced in Piron (1976), of an entity S must be testable. Such a test (sometimes also called a question or yes–no experiment) α of a property a consists of an experiment e that can be performed on S. If the experiment e gives us an expected outcome (one of the outcomes that we have considered to characterize the property), we say that the test α has succeeded, and attribute the answer *yes* to it. If the experiment e gives us other than one of the expected outcomes, we say that the test α has failed, and attribute the answer *no* to it. If α is a test for the entity S, we can consider the test that consists of performing the same experiment, but interchanging the role of *yes* and *no*. Let us denote this new test by $\tilde{\alpha}$ and call it the inverse test of α.

Each experiment e_v of our example has two possible outcomes, 1_v and 2_v. Clearly we can then make only two nontrivial tests by means of such an experiment: the test α_v, which is the test where the expected outcome is the outcome 1_v, and the test $\tilde{\alpha}_v$, which is the test where the expected outcome is the outcome 2_v. Let us denote by a_v the property tested by α_v, and by a'_v the property tested by $\tilde{\alpha}_v$.

2. *The actuality of a property.* A test α is said to be *true*, and the corresponding property a to be *actual* for the entity S, iff when we decide to perform the experiment corresponding to α, the answer *yes* comes out with certainty. Let us denote the collection of all properties of the entity S by $\mathscr{L}$.

The test α_v is *true* and the corresponding property a_v *actual* iff the entity is in state p_v, hence iff the charge q is in the direction v. Hence the property a_v can be expressed by the "proposition": "The charge q is in direction v." The test $\tilde{a}_v$ is *true* and the corresponding property a'_v *actual* iff the entity is in state p_{-v}, hence iff the charge q is in the direction $-v$. Hence the property a'_{-v} can be expressed by the "proposition": "The charge q is in direction $-v$." Here we are already confronted with the typical quantum mechanical situation, having led to so many mystifications, that the proposition used to express the property a_v is not the "negation" of the proposition used to express the property a'_v. But by means of the example we can see that there is no real mystery involved in such a situation.

3. *A physical law on the collection $\mathscr{L}$ of properties.* For two tests α and β on the entity S, we say that α is stronger than β and we write $\alpha < \beta$ iff, whenever the entity is in a state p such that α is true, then in this state p also β is true. Clearly the defined relation is a pre-order on the set of tests corresponding to the entity: (3.1) $\alpha < \alpha$; (3.2) if $\alpha < \beta$ and $\beta < \gamma$, then $\alpha < \gamma$ for any tests α, β, γ of the entity S.

If α and β are tests of the entity S such that $\alpha < \beta$ and $\beta < \alpha$, then we say that α and β are equivalent tests (and indeed we have defined here an equivalence relation). Physically when α and β are equivalent they test the same property of the entity S. Therefore we shall represent mathematically a property a of the entity S by means of the equivalence class of tests that test this property. Then the property a is *actual* for S iff there is a test $\alpha \in a$ that is *true* for S. The collection of properties $\mathscr{L}$ is then equipped with a partial order relation induced by the pre-order on the set of tests: For two properties a and b we say that $a \leq b$ iff for any tests $\alpha \in a$ and $\beta \in b$ we have $\alpha \leq \beta$. Clearly: (3.3) $a \leq a$; (3.4) if $a \leq b$ and $b \leq c$, then $a \leq c$; (3.5) if $a \leq b$ and $b \leq a$, then $a = b$; which shows that $\leq$ is a partial order relation on $\mathscr{L}$.

Let us investigate what becomes of this law on the collection of properties $\mathscr{L}$ of our example. The physical law $\alpha_v \leq \alpha_u$ is only fulfilled if $v = u$. Hence $a_v \leq a_u$ implies that $u = v$. We further have that $a'_v = a_{-v}$.

4. *The complete lattice of properties.* If $\{a_k\}$ is a collection of properties, and for every k, α_k is a test for property a_k, then we can introduce the

following new test $\pi_k\alpha_k$ that we will call the "product" of the tests α_k: "We choose one of the α_k and perform the experiment corresponding to this test α_k, then we accord to $\pi_k\alpha_k$ the answer obtained in this way." Clearly $\pi_k\alpha_k$ is *true* iff α_k is *true* for every k. Let us denote the property defined by $\pi_k\alpha_k$ by $\bigwedge_k a_k$, where a_k is the property defined by α_k. Then: (3.6) $\bigwedge_k a_k$ is *actual* iff a_k is *actual* for every k. This shows that $\bigwedge_k a_k$ is the infimum of the collection of properties $\{a_k\}$ for the partial order relation $\leq$ on $\mathscr{L}$. The collection of all tests that are never true will be denoted by 0. For an arbitrary test α we have $\alpha \cdot \tilde{\alpha} \in 0$. We can define the following trivial test τ; we do anything with the entity and give the answer yes. Clearly τ is always true. The property defined by τ will be denoted by I. For any property a we have $0 \leq a \leq I$. The collection of all properties $\mathscr{L}$ of the entity S is then a partial ordered set, with a minimal element 0 and a maximal element I, and such that for any subcollection of properties the infimum of this subcollection is also an element of $\mathscr{L}$ [see (3.6)]. Then $\mathscr{L}$ is a complete lattice [see Theorem 2.1 of Piron (1976)], the supremum for a collection of properties $\{a_k\}$ being defined by: (3.7) $\bigvee_k a_k = \wedge b$ such that $a_k \leq b$.

Let us again investigate this complete lattice structure in the case of our example. We can easily see that $\alpha_u \cdot \alpha_v$ is never "true" if u is different from v. Indeed there is no state of our entity that makes it possible to have an outcome "yes" with certainty for α_v and for α_u if u is different from v. Hence $\alpha_u \cdot \alpha_v \in 0$ for u different from v. This shows that $a_u \wedge a_v = 0$ for u different from v. This corresponds very well with the meaning of the propositional "and" because, indeed, the positive charge q is never in directions u and v when they are different. Since $a_u \leq a_v$ implies that $u = v$, it follows that $a_u \vee a_v = I$ if u is different from v. We now have all the elements of the lattice $\mathscr{L} = \{0, a_u, I\}$, and we represent it in the diagram of Figure 4.

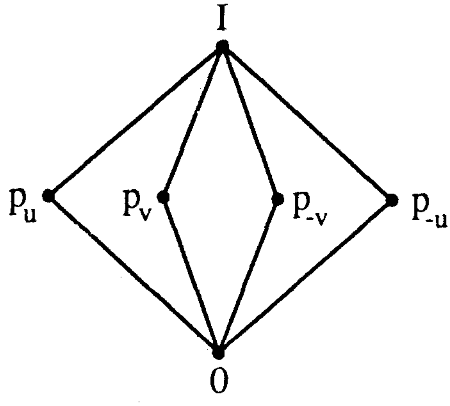

Fig. 4

5. *Orthogonal states.* Aerts' (1981, 1982) defines the following orthogonality relation $\perp$ on the set of states Σ of S: Two states p and q of an entity S are orthogonal, and we denote $p \perp q$, iff they are "eigenstates" of an experiment e with different "eigenoutcomes." This orthogonality relation is a symmetric relation which is reflexive: (3.8) if $p \perp q$, then $q \perp p$; (3.9) there is no state p orthogonal to itself.

The orthogonal states of our example are p_u and p_{-u} for every point u.

6. *Orthogonal properties.* The orthogonality relation on the states of the entity defines in a natural way an orthogonality relation on the properties of the entity: Two properties a and b are orthogonal, and we denote $a \perp b$, iff every state p that makes a actual is orthogonal to every state q that makes b actual. This orthogonality relation has the following properties: (3.10) if $a \perp b$, then $b \perp a$; (3.11) if $a \perp b$ and $c \leq a$ and $d \leq b$, then $c \perp d$; (3.12) if $a \perp b$, then $a \wedge b = 0$.

The orthogonal properties of our example are a_u and a_{-u} for every point u.

3.2. The Non-Boolean Character of the Lattice of Properties of Our Mechanistic Macroscopic Example

Let us show the non-Boolean nature of this lattice, by considering two different points u and v of the sphere, such that $-u$ is also different from v. We have $a_u \vee a_{-u} = I$; hence

$$(a_u \vee a_{-u}) \wedge a_v = I \wedge a_v = a_v \tag{1}$$

We also have $a_u \wedge a_v = a_{-u} \wedge a_v = 0$; hence

$$(a_u \wedge a_v) \vee (a_{-u} \wedge a_v) = 0 \vee 0 = 0 \tag{2}$$

3.3. Construction of the Lattice of Properties by Means of the Orthogonality Relation

It would be very interesting to be able to construct indirectly from the graph of the dynamics of measurements a lattice which is isomorphic to the lattice of properties of the entity. This problem has been considered in Grib and Zapatrin (1990), by using an orthogonality relation on the set of states of the entity. Of course we have to make sure that the lattice that we get by this mathematical construction is indeed isomorphic with the original lattice of properties, if we want to use it for drawing physical conclusions about the nature of the entity (classical or nonclassical or other types of physical conclusions). Let us analyze how this can be done.

1. *The Cartan representation.* Every property a of the lattice of properties can be related to the collection of states $\mu(a)$ that make this property actual. This relation induces a map $\mu: \mathscr{L} \to \mathscr{P}(\Sigma)$ that can easily be shown to be an injective lattice morphism (Aerts, 1981; Piron, 1990). The property I is mapped onto Σ, and the property 0 onto the element $\varnothing$. Hence if we consider $\mu(\mathscr{L})$ which is included in $\mathscr{P}(\Sigma)$ (the power set of the set of states Σ), then it is isomorphic to $\mathscr{L}$ (μ is an isomorphism onto its image). From this result it follows that we can concentrate on an attempt to construct this lattice $\mu(\mathscr{L})$ as the collection of subsets of Σ. We will do this by using the orthogonality relation that naturally exists on the set of states. Indeed, without considering the directly constructed lattice of properties of the entity S, the orthogonality relation $\perp$ on the set Σ of states of the entity S gives us a way to construct a complete lattice. Let us explain this construction.

2. *The construction of a complete lattice by means of the orthogonality relation on the set of states.* Grib and Zapatrin (1990) emphasized that sometimes even in macroscopic situations one should use "negative" logic when only a negative answer on some question can give exact information about the state of the entity. This can be done due to the orthogonality relation on the set of states. So properties are defined by "double negation," or in the mathematical sense by the biorthogonals of sets of states. Let us therefore introduce the following definitions: For a subset A of Σ, $A^\perp = \{p \mid p \in \Sigma, \text{ and } p \perp q \text{ for all } q \in A\}$ is the orthogonal of this subset. We will call $A^{\perp\perp}$ the biorthogonal of the subset A. As follows from the theorem below, this biorthogonal has the properties of a closure. Therefore we also denote it by $cl(A)$, and call a subset of Σ "closed" iff $cl(A) = A$. From Birkhoff (1967) it follows that we have the following properties for this biorthogonal relation: (3.13) if $A \subseteq B$, then $B^\perp \subseteq A^\perp$; (3.14) $A \subseteq cl(A)$; (3.15) $A^\perp = A^{\perp\perp\perp}$; (3.16) if $A \subseteq B$, then $cl(A) \subseteq cl(B)$; (3.17) $cl(cl(A)) = cl(A)$.

From these properties one can easily prove the following theorem.

Theorem. If we have an entity S with a set of states Σ and an orthogonality relation $\perp$ on Σ, then the collection of closed subsets of the set of all states Σ, denoted by L, forms a complete lattice, with partial order relation the set-theoretic inclusion, infimum the set-theoretic intersection. The supremum of a collection of elements is the smallest closed subset that contains all these elements.

This method of constructing the lattice leads to an isomorphic lattice of properties if certain axioms are satisfied (Aerts, 1981; Piron, 1990, 1.7.4).

3.4. Construction of the Lattice of Closed Subsets for Our Example

Let us make the proposed construction for our example. We have $\Sigma = \{p_u \mid u \text{ is an arbitrary point of the sphere}\}$. We have mentioned already the orthogonality relation on Σ, namely $p_u \perp p_{-u}$. Then $p_u^{\perp\perp} = p_{-u}^{\perp} = p_u$, which shows that $\{p_u\}$ is an element of L for any point u of the sphere. On the other hand, for example, for u different from v we have $\{p_u, p_v\}^{\perp\perp} = 0^{\perp} = \Sigma$ since there are no states orthogonal to p_u and p_v. The same can be seen for all the other elements of $\mathscr{P}(\Sigma)$. This shows that this way of constructing the lattice gives us $L = \{0, \{p_u\}, \Sigma\}$, which is a lattice isomorphic to $\mathscr{L}$. From the analysis that we made for the lattice $\mathscr{L}$ we see that this lattice also is non-Boolean.

4. CONCLUSION

We have presented a macroscopic mechanistic entity that gives rise to a non-Boolean lattice of properties that can easily be seen to be isomorphic to the lattice of properties of the spin of a spin-1/2 quantum entity. This coincides with the result found in Aerts (1985, 1986, 1987), where it is shown that also the probability model of the mechanistic macroscopic entity is isomorphic to the probability model of the spin of a spin-1/2 quantum entity. For the example that we have considered here, the two methods of constructing the lattice of properties of the macroscopic mechanistic entity, the direct method and the mathematical method using the orthogonality relation on the set of states, lead to lattices that are isomorphic. It can easily be shown that in the case of a quantum entity described in a complex Hilbert space, these two methods will also lead to lattices that are isomorphic. Indeed both methods deliver the well-known lattice of all closed subspaces of the Hilbert space. However, for a general experiment situation, the two methods may not necessarily lead to the same complete lattice of properties. The fact that both methods lead to the same lattice of properties is related to the existence of an orthocomplementation on this lattice of properties. In a following paper we will analyze why this is so, and the physical meaning of the fact that for quantum entities both methods lead to the "good" lattice of properties. We shall also investigate the lattice of "operational propositions" as defined in Randall and Foulis (1983).

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